Cartan Maps and Projective Modules

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Abstract. Let R be a commutative ring, π be a finite group, $R\pi$ be the group ring of π over R. Theorem 1. If R is a commutative artinian ring and π is a finite group. Then the Cartan map $c : K_0(R\pi) \to G_0(R\pi)$ is injective. Theorem 2. Suppose that R is a Dedekind domain with $\operatorname{char} R = p > 0$ and π is a p-group. Then every finitely generated projective $R\pi$-module is isomorphic to $F \oplus \mathcal{A}$ where F is a free module and $\mathcal{A}$ is a projective ideal of $R\pi$. Moreover, R is a principal ideal domain if and only if every finitely generated projective $R\pi$-module is isomorphic to a free module. Theorem 3. Let R be a commutative noetherian ring with total quotient ring K, A be an R-algebra which is a finitely generated R-projective module. Suppose that I is an ideal of R such that R/I is artinian. Let $\{\mathcal{M}_1, \ldots, \mathcal{M}_n\}$ be the set of all maximal ideals of R containing I. Assume that the Cartan map $c_i : K_0(A/\mathcal{M}_iA) \to G_0(A/\mathcal{M}_iA)$ is injective for all $1 \leq i \leq n$. If P and Q are finitely generated A-projective modules with $KP \simeq KQ$, then $P/IP \simeq Q/IQ$.

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§1. Introduction

Throughout this note, $R\pi$ denotes the group ring where π is a finite group and R is a commutative ring; all the modules we consider are left modules. The present article arose from an attempt to understand the following theorem of Swan.

Theorem 1.1 (Swan [Sw1]) *Let R be a Dedekind domain with quotient field K and π be a finite group. Assume that* $\operatorname{char} R = 0$ *and no prime divisor of $|\pi|$ is a unit in R. If P is a finitely generated projective $R\pi$-module, then $K \otimes_R P$ is a free $K\pi$-module and P is isomorphic to $F \oplus \mathcal{A}$ where F is a free $R\pi$-module and $\mathcal{A}$ is a left ideal of $R\pi$. Moreover, for any non-zero ideal I of R, we may choose $\mathcal{A}$ such that $I + (R \cap \mathcal{A}) = R$.*

Several alternative approaches to the proof of some parts of Theorem 1.1 were proposed; see, for examples, [Ba1], [Gi], [Ri2], [Ha], [Gr, page 20], [Sw3, page 57, Theorem 4.2]; also see [Sw2, page 171, Theorem 11.2]. Using the injectivity of the Cartan map (see Definition 2.4), Bass recast a crucial step of the proof of Theorem 1.1 as follows.

Theorem 1.2 (Bass [Ba1, Theorem 1]) *Let R be a commutative noetherian ring with total quotient ring K and denote by m-spec(R) the space of all the maximal ideals of R (under Zariski topology) with $d = \dim(m\text{-}\operatorname{spec}(R))$. Let A be an R-algebra which, as an R-module, is a finitely generated projective R-module. Suppose that P is a finitely generated projective A-module satisfying that* (i) *$K \otimes_R P$ is a free KA-module of rank r, and* (ii) *the Cartan map $c_{\mathcal{M}} : K_0(A/\mathcal{M}A) \to G_0(A/\mathcal{M}A)$ is injective for any $\mathcal{M} \in m$-spec(R). Then P is isomphic to $F \oplus Q$ where F is a free A-module of rank r' and $Q/\mathcal{M}Q$ is a* rank d' *free module over $A/\mathcal{M}A$ for any $\mathcal{M} \in m$-spec(R) with $d' = min\{d, r\}$ and $r' = r - d'$.*

Note that the assumption about the Cartan map in Theorem 1.2 is valid when $A = R\pi$ where π is a finite group, thanks to the following theorem of Brauer and Nesbitt.

Theorem 1.3 (Brauer and Nesbitt [BN1; BN2; Br; CR, page 442]) *Let k be a field, π be a finite group. Then the Cartan map $c : K_0(k\pi) \to G_0(k\pi)$ is injective.*

It is known that the Cartan map $c : K_0(A) \to G_0(A)$ is an isomorphism if the (left) global dimension of A is finite [Ei, Proposition 21; Sw2, page 104, Corollary 4.7]. However, it is possible that the global dimension of A is infinite while the Cartan map is injective. By Lemma 2.11 the global dimension of the group ring $k\pi$ (k is a field) is infinite if $\operatorname{char} k = p > 0$ and $p \mid |\pi|$. Thus Theorem 1.3 provides plenty of

such examples. For examples other than the group rings, see [EIN, Section 5], [La3, Example 5.76], [BFVZ] and also [La1, Theorem 2.4; St].

In this article we will prove the following result which generalizes Theorem 1.3.

Theorem 1.4 *Let R be a commutative artinian ring and π be a finite group. Then the Cartan map $c : K_0(R\pi) \to G_0(R\pi)$ is injective.*

The main idea of the proof of Theorem 1.4 is to use the Frobenius functors as in Lam's paper [La1]. For a generalization of this theorem, see Theorem 4.3.

We will also study a variant of Theorem 1.1, i.e. finitely generated $R\pi$-projective modules where R is a Dedekind domain with $\operatorname{char} R = p > 0$. One of our results is the following (see Theorem 3.1 and Theorem 3.3).

Theorem 1.5 *Let R be a Dedekind domain with quotient field K such that* $\operatorname{char} R = p > 0$. *Let π be a finite group with $p \mid |\pi|$, and π_p be a p-Sylow subgroup of π.*

(1) *Let M be a finitely generated $R\pi$-module. Then*

  M is a projective $R\pi$-module,
  $\Leftrightarrow$ *The restriction of M to $R\pi_p$ is a projective $R\pi_p$-module,*
  $\Leftrightarrow$ *The restriction of M to $R\pi'$ is a projective $R\pi'$-module where π' is any elementary abelian subgroup of π_p.*

(2) *If π is a p-group and P is a finitely generated projective $R\pi$-module, then $K \otimes_R P$ is a free $K\pi$-module and P is isomorphic to $F \oplus \mathcal{A}$ where F is a free module and $\mathcal{A}$ is a projective ideal of $R\pi$. Moreover, for any non-zero ideal I of R, we may choose $\mathcal{A}$ such that $I + (R \cap \mathcal{A}) = R$.*

In the situation of Part (2) of the above theorem, we will show in Theorem 3.5 that R is a principal ideal domain if and only if every finitely generated $R\pi$-projective module is free. For more cases, see Lemma 3.6, Lemma 4.6 and Lemma 4.9.

Terminology and notations. For the sake of brevity, a projective module over a ring A will be called an A-projective module (or simply A-projective). A projective ideal $\mathcal{A}$ of A is a left ideal of the ring A such that $\mathcal{A}$ is A-projective. An A-module M is called indecomposable if $M \simeq M_1 \oplus M_2$ implies either $M_1 = 0$ or $M_2 = 0$; similarly for indecomposable projective modules. If A is a ring we will denote by $\operatorname{rad}(A)$ the Jacobson radical of A. If A is an R-algebra where R is a commutative ring with total quotient K, we denote $KA := K \otimes_R A$, $KM := K \otimes_R M$ if M is an A-module; similarly, $R_{\mathcal{M}}$ denotes the localization of R at the maximal ideal $\mathcal{M}$ and $M_{\mathcal{M}} := R_{\mathcal{M}} \otimes_R M$ if M is an A-module.

An $R\pi$-lattice M is a finitely generated $R\pi$-module which is an R-projective module as an R-module (see Definition 2.6). Two $R\pi$-lattices M and N belong to the same genus if $R_{\mathcal{M}} \otimes_R M$ is isomorphic to $R_{\mathcal{M}} \otimes_R N$ for any maximal ideal $\mathcal{M}$ of R [CR, page 643].

If M is an $R\pi$-module and π' is a subgroup of π, then we may regard M as an $R\pi'$-module through the ring homomorphism $R\pi' \to R\pi$; such an $R\pi'$-module is called the restriction of M to $R\pi'$ and is denoted by $M_{\pi'}$. On the other hand, if N is an $R\pi'$-module and π' is a subgroup of π, then the $R\pi$-module $R\pi \otimes_{R\pi'} N$ is called the induced module of N and is denoted by N^{π}. For details, see [CR, page 228].

We say that R is a local ring or a semilocal ring if R is a commutative noetherian ring with a unique or only finitely many maximal ideals; a local ring is denoted by $(R, \mathcal{M})$ where $\mathcal{M}$ is the maximal ideal of R. To avoid possible confusion we will not define the notion of a non-commutative semilocal ring [Sw2, page 170; La2, page 311], because it appears only once in Example 3.8 of this article. A (possibly non-commutative) ring R is called quasi-local if all the non-unit elements form a two-sided ideal [Sw2, page 77]. All the A-projective modules we consider are finitely generated, unless otherwise specified. If M is an A-module, the direct sum of n copies of it is denoted by $M^{(n)}$.

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§2. The Cartan map

Recall the definitions of the Grothendieck groups $K_0(A)$ and $G_0(A)$. Let A be a ring. Then $K_0(A)$ is the abelian group defined by generators $[P]$ where P is a finitely generated A-projective module, with relations $[P] = [P'] + [P'']$ whenever there is a short exact sequence of projective A-modules $0 \to P' \to P \to P'' \to 0$. In a similar way, if A is a left noetherian ring, then $G_0(A)$ is the abelian group defined by generators $[M]$ where M is a finitely generated A-module, with relations $[M] = [M'] + [M'']$ whenever an exact sequence $0 \to M' \to M \to M'' \to 0$ exists. For details, see [Sw3, Chapter 1].

Definition 2.1 ([Sw2, page 86]) Let A be a ring, I be a two-sided ideal of A. We say that A is I-complete if the natural map $A \to \varprojlim_{n \in \mathbb{N}} A/I^n$ is an isomorphism.

Lemma 2.2 ([Sw2, page 89, Theorem 2.26]) *If I is a two-sided ideal of a ring A such that A is I-complete, then there is a one-to-one correspondence between the isomor-*

phism classes of finitely generated A-projective modules and the isomorphism classes of finitely generated A/I-projective modules given by $P \rightsquigarrow P/IP$.

Lemma 2.3 *Let A be a left artinian ring with Jacobson radical J. Then $K_0(A)$ and $G_0(A)$ are free abelian groups of the same rank. In fact, it is possible to find finitely generated indecomposable projective A-modules $P_1, P_2, \ldots, P_n$ and simple A-modules $M_1, M_2, \ldots, M_n$ such that, for $1 \le i \le n$, $M_i \simeq P_i/JP_i$ and $K_0(A) = \bigoplus_{1\le i\le n} \mathbb{Z}\cdot[P_i]$, $G_0(A) = \bigoplus_{1\le i\le n} \mathbb{Z}\cdot[M_i]$. Moreover, each P_i is a left ideal generated by some idempotent element of A.*

Proof. Since J is a nilpotent ideal [La2, page 56], A is J-complete. On the other hand, if M is a simple A-module, then $J \cdot M = 0$ [La2, page 54]; thus the family of simple A-modules is identical to that of simple A/J-modules.

Note that A/J is left artinian and $\mathrm{rad}(A/J) = 0$. It is semisimple by the Artin-Wedderburn Theorem [La2, page 57]. Since any A/J-module is projective [La2, page 29], a simple A/J-module is an indecomposable A/J-projective module. If Q is a finitely generated indecomposable A/J-projective module, then $Q \oplus Q' \simeq (A/J)^{(t)}$ for some module Q' and some integer t. By the Krull-Schmidt-Azumaya Theorem [CR, page 128], Q is isomorphic to some minimal left ideal of A/J. It follows that every finitely generated indecomposable A/J-projective module is isomorphic to a minimal left ideal of A/J (which is generated by some idempotent of A/J). Thus the family of simple A/J-modules is identical to that of finitely generated indecomposable A/J-projective modules.

Apply the correspondence of Lemma 2.2. Since A is J-complete, any idempotent in A/J can be lifted to one in A [Sw2, page 86, Proposition 2.19], which gives rise to an indecomposable A-projective module. ■

Definition 2.4 Let A be a left artinian ring. The Cartan map $c : K_0(A) \to G_0(A)$ is defined as follows. For any finitely generated A-projective module P, find a Jordan-Hölder composition series of P : $M_0 = P \supset M_1 \supset M_2 \supset \cdots \supset M_t = \{0\}$, where each M_i/M_{i+1} is a simple A-module. Define $c([P]) = \sum_{0\le i\le t-1}[M_i/M_{i+1}] \in G_0(A)$. It is easy to see that c is a well-defined group homomorphism.

By Lemma 2.3, write $K_0(A) = \bigoplus_{1\le i\le n} \mathbb{Z}\cdot[P_i]$, $G_0(A) = \bigoplus_{1\le i\le n} \mathbb{Z}\cdot[M_i]$. If $c([P_i]) = \sum_{1\le j\le n} a_{ij}[M_j]$ where $a_{ij} \in \mathbb{Z}$, the matrix $(a_{ij})_{1\le i,j\le n}$ is called the Cartan matrix. Clearly the Cartan map is injective if and only if $\det(a_{ij}) \neq 0$.

In general, the Cartan map $c : K_0(A) \to G_0(A)$ may be defined for a left noetherian ring A by sending $[P] \in K_0(A)$ (where P is a finitely generated A-projective module) to $[P] \in G_0(A)$ by regarding P as a finitely generated A-module. As noted before, if A is a left noetherian ring with finite global dimension, then the Cartan map $c : K_0(A) \to$

$G_0(A)$ is an isomorphism [Sw2, page 104]. In this article we will restrict our attention only to Cartan maps of left artinian rings.

Lemma 2.5 *Let A be a left artinian ring with Jacobson radical J. Then A contains finitely many indecomposable projective ideals, $P_1, P_2, \ldots, P_n$, satisfying the following properties,*

(i) $P_i \not\simeq P_j$ *if* $i \neq j$*;*

(ii) *each P_i is generated by an idempotent element of A;*

(iii) *every finitely generated A-projective module is isomorphic to $\bigoplus_{1 \leq i \leq n} P_i^{(m_i)}$ for some non-negative integers m_i;*

(iv) $\{P_i/JP_i : 1 \leq i \leq n\}$ *forms the family of all the isomorphism classes of simple A-modules. In fact, P_i is the projective cover of P_i/JP_i.*

Proof. The proofs of (i), (ii) and (iii) are implicit in the proof of Lemma 2.3. As to the definition of projective covers, see [Sw2, page 88]. The proof of (iv) follows from [Sw2, page 89, Corollary 2.25]. ■

For the proof of Theorem 1.4 recall the definitions of $G_0^R(R\pi)$ and Frobenius functors. Note that the definition of Frobenius functors in Definition 2.7 is that given in [Sw3] and is slightly different from that in [La1].

Definition 2.6 ([Sw3, page 2]) Let R be a commutative ring, A be an R-algebra which is a finitely generated R-module. Define $G_0^R(A)$ to be the abelian group with generators $[M]$ where M is a finitely generated A-module which is R-projective as an R-module, with relations $[M] = [M'] + [M'']$ whenever there is a short exact sequence of A-modules $0 \to M' \to M \to M'' \to 0$ such that M', M, M'' are R-projective as R-modules. Note that $G_0^R(R\pi)$ is a commutative ring if π is a finite group [Sw3, page 7].

Definition 2.7 ([La1; Sw3, page 15]) Let π be a finite group, $\underline{\mathrm{Grp}}_\pi$ be the category whose objects are all the subgroups of π with morphisms $\hom(\pi_1, \pi_2)$ consisting of the unique injection if $\pi_1 \subset \pi_2 \subset \pi$ with the understanding that $\hom(\pi_1, \pi_2) = \emptyset$ if $\pi_1 \not\subset \pi_2$. Let $\underline{\mathrm{Ring}}$ be the category of commutative rings. A Frobenius functor consists of the following data,

(i) for each subgroup π' of π, there corresponds a commutative ring $F(\pi')$,

(ii) for subgroups $\pi_1 \subset \pi_2 \subset \pi$ and the injection $i : \pi_1 \to \pi_2$, there exist the ring homomorphism $i^* : F(\pi_2) \to F(\pi_1)$ and the additive group homomorphism $i_* : F(\pi_1) \to F(\pi_2)$ satisfying the properties that $i^* : \underline{\mathrm{Grp}}_\pi \to \underline{\mathrm{Ring}}$ is a contravariant functor and i_* from finite groups to abelian groups is a covariant functor,

(iii) (Frobenius identity) for each injection $i : \pi_1 \to \pi_2$, if $x \in F(\pi_1)$, $y \in F(\pi_2)$, then $i_*(x) \cdot y = i_*(x \cdot (i^* y))$.

It is not difficult to see that $\pi' \mapsto G_0^R(R\pi')$ is a Frobenius functor where R is a commutative ring and $G_0^R(R\pi')$ is defined in Definition 2.6.

Definition 2.8 Given a finite group π and a Frobenius functor $F : \underline{\mathrm{Grp}}_\pi \to \underline{\mathrm{Ring}}$, a Frobenius module M over F consists of the data

(i) for each subgroup π' of π, there corresponds an $F(\pi')$-module $M(\pi')$;

(ii) for each injection $i : \pi_1 \to \pi_2$, there exist the contravariant additive functor $i^* : M(\pi_2) \to M(\pi_1)$ and the covariant additive functor $i_* : M(\pi_1) \to M(\pi_2)$ such that if $x \in F(\pi_2)$, $u \in M(\pi_2)$, then $i^*(x \cdot u) = i^*(x) \cdot i^*(u)$;

(iii) for any injection $i : \pi_1 \to \pi_2$ and $x \in F(\pi_1)$, $v \in M(\pi_2)$, then $i_*(x) \cdot v = i_*(x \cdot i^*(v))$; if $y \in F(\pi_2)$, $u \in M(\pi_1)$, then $y \cdot i_*(u) = i_*((i^* y) \cdot u)$.

Let R be a commutative ring, π be a finite group. Let F be the Frobenius functor defined by $\pi' \mapsto G_0^R(R\pi')$. It is easy to show that $\pi' \mapsto G_0(R\pi')$ and $\pi' \mapsto K_0(R\pi')$ are Frobenius modules over F.

The morphism of Frobenius modules over a given Frobenius functor can be defined in an obvious way. For details, see [Sw3, pages 16–18]. If M_1 and M_2 are Frobenius modules over a Frobenius functor F and $\varphi : M_1 \to M_2$ is a morphism over F, then $\mathrm{Ker}(\varphi)$ and $\mathrm{Coker}(\varphi)$, defined in the obvious way, are also Frobenius modules over F.

If R is a commutative artinian ring, the Cartan map of Definition 2.4 defined by $K_0(R\pi') \to G_0(R\pi')$ is a morphism of Frobenius modules over the Frobenius functor $\pi' \to G_0^R(R\pi')$.

Definition 2.9 ([Sw3, pages 22–23]) Let π be a finite group, $\mathcal{C}$ be a class of certain subgroups of π. If $F : \underline{\mathrm{Grp}}_\pi \to \underline{\mathrm{Ring}}$ is a Frobenius functor and M is a Frobenius module over F. We define

$$F(\pi)_{\mathcal{C}} = \sum_{\pi' \in \mathcal{C}} i_*(F(\pi')), \quad M(\pi)_{\mathcal{C}} = \sum_{\pi' \in \mathcal{C}} i_*(M(\pi')),$$
$$M(\pi)^{\mathcal{C}} = \bigcap_{\pi' \in \mathcal{C}} \mathrm{Ker}\{i^* : M(\pi) \to M(\pi')\}.$$

It can be shown that $F(\pi)_{\mathcal{C}}$ is an ideal of $F(\pi)$, $M(\pi)_{\mathcal{C}}$ and $M(\pi)^{\mathcal{C}}$ are submodules of $M(\pi)$ over $F(\pi)$, both of $M(\pi)/M(\pi)_{\mathcal{C}}$ and $M(\pi)^{\mathcal{C}}$ are modules over $F(\pi)/F(\pi)_{\mathcal{C}}$, (see [Sw3, pages 22-23, Lemma 2.6 and Lemma 2.7]).

Now we turn to the proof of Theorem 1.4. Our proof is an adaptation of the proof in [Sw3, page 36, Theorem 2.20].

Suppose that R is a commutative artinian ring and π is a finite group. We will show that the Cartan map $c_\pi : K_0(R\pi) \to G_0(R\pi)$ is injective.

Step 1. We claim that if $c_{\pi'}$ is injective for any cyclic subgroup π' of π, then c_π is injective for the group π.

Consider the Frobenius functor $F : \underline{\mathrm{Grp}}_\pi \to \underline{\mathrm{Ring}}$ defined by $F(\pi') = G_0^R(R\pi')$ where π' is any subgroup of π. Note that the Cartan map $c_{\pi'} : K_0(R\pi') \to G_0(R\pi')$ is a morphism of Frobenius modules $K_0(R\pi) \to G_0(R\pi)$ over the Frobenius functor F. Define a Frobenius module by $M(\pi') = \mathrm{Ker}\{c_{\pi'} : K_0(R\pi') \to G_0(R\pi')\}$. Note that $M(\pi') = 0$ if π' is a cyclic subgroup by the assumption at the beginning of this step.

Let $\mathcal{C}$ be the class of all the cyclic subgroups of π. Thus $M(\pi)^{\mathcal{C}} = M(\pi)$ since $M(\pi') = 0$ if π' is cyclic.

Let $|\pi| = n$. Then $n^2 \cdot (G_0^R(R\pi)/G_0^R(R\pi)_{\mathcal{C}}) = 0$ by Artin's induction theorem [Sw3, page 24, Corollary 2.12]. Since $M(\pi)^{\mathcal{C}}$ is a module over $G_0^R(R\pi)/G_0^R(R\pi)_{\mathcal{C}}$, it follows that $n^2 \cdot M(\pi)^{\mathcal{C}} = 0$ by [Sw3, page 23, Lemma 2.10].

As $M(\pi)^{\mathcal{C}} = M(\pi)$ and $M(\pi)$ is a subgroup of $K_0(R\pi)$ which is a free abelian group of finite rank by Lemma 2.3, we find that $M(\pi)^{\mathcal{C}}$ is a torsion subgroup of $K_0(R\pi)$. It follows that $M(\pi)^{\mathcal{C}} = 0$. Thus $c_\pi : K_0(R\pi) \to G_0(R\pi)$ is injective.

Note that the above arguments was formalized in [La1, Corollary 3.5].

Step 2. It remains to show that $c_\pi : K_0(R\pi) \to G_0(R\pi)$ is injective if π is a cyclic group.

Without loss of generality, we may assume that R is a commutative artinian local ring. Write $R = (R, \mathcal{M})$ where $\mathcal{M}$ is the maximal ideal of R and $k = R/\mathcal{M}$ is the residue field.

Let $\pi = \langle \sigma \rangle$ be a cyclic group of order m. We may write $k\pi = k[\sigma] \simeq k[X]/\langle X^m - 1 \rangle$ where $k[X]$ is the polynomial ring. Note that $\mathrm{rad}(R) = \mathcal{M}$ and $\mathrm{rad}(R) \cdot R\pi \subset \mathrm{rad}(R\pi)$ (see, for examples, [La2, page 74, Corollary 5.9; Sw2, page 170, Lemma 11.1]). Thus

$$R\pi/\mathrm{rad}(R\pi) \simeq \frac{R\pi/\mathcal{M} \cdot R\pi}{\mathrm{rad}(R\pi)/\mathcal{M} \cdot R\pi} \simeq k\pi/\mathrm{rad}(k\pi) \simeq k[X]/\langle f(X) \rangle$$

with $f(X) = \prod_{1 \le i \le t} f_i(X)$ where $f_1(X), \ldots, f_t(X)$ are all the distinct monic irreducible factors of $X^m - 1$ in $k[X]$.

It follows that $S_i = k[X]/\langle f_i(X) \rangle$, $1 \le i \le t$, are all the simple modules over $k[X]/\langle f(X) \rangle \simeq R\pi/\mathrm{rad}(R\pi)$. By Lemma 2.5, $S_1, \ldots, S_t$ are all the non-isomorphic simple $R\pi$-modules and their projective covers $P_1, \ldots, P_t$ are all the non-isomorphic indecomposable $R\pi$-projective modules. Consequently, $K_0(R\pi) = \bigoplus_{1 \le i \le t} \mathbb{Z} \cdot [P_i]$ and $G_0(R\pi) = \bigoplus_{1 \le i \le t} \mathbb{Z} \cdot [S_i]$. We will consider the Cartan map $c_\pi : K_0(R\pi) \to G_0(R\pi)$.

Write $J = \mathrm{rad}(R\pi)$. For $1 \le i \le t$, consider the filtration $P_i \supset JP_i \supset J^2P_i \supset \cdots \supset J^sP_i = \{0\}$ (note that J is a nilpotent ideal). Each quotient module $J^jP_i/J^{j+1}P_i$ can be regarded as a module over $R\pi/J \simeq k[X]/\langle f(X)\rangle$. Note that $\bar{f}_i \cdot J^jP_i/J^{j+1}P_i = 0$ because $0 = \bar{f}_i \cdot S_i \simeq \bar{f}_i \cdot P_i/JP_i$ (remember that $R\pi$ is a commutative ring as π is cyclic). Thus $J^jP_i/J^{j+1}P_i$ becomes a module over $k[X]/\langle f_i(X)\rangle$. It follows that the only simple $R\pi$-module which may arise as a Jordan-Hölder composition factor of P_i is $S_i = k[X]/\langle f_i(X)\rangle$. We conclude that $c_\pi([P_i]) = a_i[S_i]$ for some positive integer a_i. Hence the determinant of the Cartan matrix is non-zero. ∎

Example 2.10 Let R be a commutative artinian ring, π be a finite group. By Lemma 2.5, every finitely generated projective $R\pi$-module is a direct sum of projective ideals generated by some idempotent of $R\pi$. If P and Q are finitely generated $R\pi$-projective modules, we will show that $P \simeq Q$ if and only if P and Q have the same composition factors. For, if $[P] = [Q]$ in $G_0(R\pi)$, then $c([P]-[Q]) = 0$ where $c : K_0(R\pi) \to G_0(R\pi)$ is the Cartan map. By Theorem 1.4, $[P] = [Q]$ in $K_0(R\pi)$. Thus $P \oplus F \simeq Q \oplus F$ for some free $R\pi$-module F of finite rank. By the Krull-Schmidt-Azumaya Theorem [CR, page 128], we find that $P \simeq Q$.

On the other hand, let A be a left artinian ring such that the Cartan map $c : K_0(A) \to G_0(A)$ is not injective (such an artinian ring does exist by [BFVZ, Lemma 2]). By Lemma 2.3, choose indecomposable A-projective modules $P_1, P_2, \ldots, P_n$ and simple A-modules $M_1, M_2, \ldots, M_n$ such that, for $1 \le i \le n$, $M_i \simeq P_i/JP_i$. Then there is some $1 \le i \le n$ such that M_j arises in the composition factor of P_i for some $j \ne i$; otherwise, the determinant of the Cartan matrix would be positive. In general the Cartan matrix is a diagonal matrix (as in the proof of the Theorem 1.4) if and only if $Hom_A(P_i, P_j) = 0$ for any $1 \le i, j \le n$ with $i \ne j$ by [La2, page 325, Proposition (21.19)].

The following lemma is a folklore among experts (see, for example, [La3, page 190]). We include it here for completeness.

Lemma 2.11 *Let k be a field, π be a finite group.*

(i) *If $\mathrm{char}\, k = 0$ or $\mathrm{char}\, k = p > 0$ with $p \nmid |\pi|$, then the global dimension of $k\pi$ is zero.*

(ii) *If $\mathrm{char}\, k = p > 0$ with $p \mid |\pi|$, then the global dimension of $k\pi$ is infinite.*

Proof. (i) $k\pi$ is semisimple by Maschke's Theorem. Thus every $k\pi$-module is projective [La2, page 29].

(ii) $k\pi$ is right self-injective by [La3, page 420, Exercise 14]. Hence it is right Kasch [La3, page 411]. By [La3, page 189, Corollary 5.74] the global dimension of $k\pi$ is either zero or infinite.

Now suppose that $\operatorname{char} k = p > 0$ and $p \mid |\pi|$. Once we find a $k\pi$-module which is not projective, then we are done (because of the assertion of the above paragraph).

Define $u = \sum_{\sigma \in \pi} \sigma \in k\pi$. Then $u^2 = 0$ and u belongs to the center of $k\pi$.

Write $I = k\pi \cdot u$, the ideal generated by u. We claim that $k\pi / I$ is not a projective $k\pi$-module.

Otherwise, I is a direct summand of $k\pi$. It follows that $I = k\pi \cdot e$ for some idempotent e of $k\pi$. Write $e = \alpha u$ where $\alpha \in k\pi$. Then $e = e^2 = (\alpha u)(\alpha u) = \alpha^2 u^2 = 0$. This is impossible. ■

§3. Projective modules

Theorem 3.1 *Let R be a Dedekind domain with* $\operatorname{char} R = p > 0$. *Let π be a finite group, M be an $R\pi$-module. Assume that $p \mid |\pi|$ and choose a p-Sylow subgroup π_p of π. Then M is an $R\pi$-projective module. $\Leftrightarrow$ The restriction of M to $R\pi_p$ is an $R\pi_p$-projective module. $\Leftrightarrow$ The restriction of M to $R\pi'$ is an $R\pi'$-projective module where π' is any elementary abelian subgroup of π_p.*

Proof. Suppose M is an $R\pi_p$-projective module. We will show that M is a projective module over $R\pi$. Since $[\pi : \pi_p]$ is a unit in R, it follows that M is (π, π_p)-projective and M is a direct summand of $(M_{\pi_p})^\pi$ by [CR, page 452, Proposition 19.5] (where M_{π_p} is the restriction of M to $R\pi_p$, and $(M_{\pi_p})^\pi := R\pi \otimes_{R\pi_p} (M_{\pi_p})$).

Since M_{π_p} is an $R\pi_p$-projective module, it follows that $(M_{\pi_p})^\pi$ is an $R\pi$-projective module. So is its direct summand M.

Now assume that M is an $R\pi'$-projective module for all elementary abelian p-group π' of π_p. By [Ch, Corollary 1.1], M is an $R\pi_p$-projective module.

Note that, by a theorem of Rim [Ri1, Proposition 4.9], a module M is $R\pi$-projective if and only if so is it when restricted to all the Sylow subgroups of π. But the situation of our theorem requires that $\operatorname{char} R = p > 0$; thus only a p-Sylow subgroup is sufficient to guarantee the projectivity over $R\pi$. ■

Remark. If π is a p-group, recall the definition of the Thompson subgroup of π, which is denoted by $J(\pi)$ [Is, page 202]: $J(\pi)$ is the subgroup of π generated by all the elementary abelian subgroups of π.

With the definition of $J(\pi)$, we may rephrase Chouinard's theorem [Ch, Corollary 1.1] as follows: Let π be a p-group and M be an $R\pi$-module where R is any commutative ring. Then M is an $R\pi$-projective module if and only if so is its restriction to the group

ring of $J(\pi)$ over R. Similarly, Theorem 3.1 may be formulated via the Thompson subgroup of π_p.

Recall the following well-known lemma, which will be used in the sequel.

Lemma 3.2 ([Ba1, Lemma 2.4; Sw3, page 13]) *Let A be a ring, I be a two-sided ideal of A with $I \subset \operatorname{rad}(A)$. If P and Q are finitely generated A-projective modules satisfying that $P/IP \simeq Q/IQ$, then $P \simeq Q$.*

Theorem 3.3 *Let p be a prime number, π be a p-group, and R be a Dedekind domain with quotient field K such that $\operatorname{char} R = p$. If P is a finitely generated $R\pi$-projective module, then KP is a free $K\pi$-module, and $P \simeq F \oplus \mathcal{A}$ where F is a free $R\pi$-module and $\mathcal{A}$ is a projective ideal of $R\pi$. Moreover, for any non-zero ideal I of R, we may choose $\mathcal{A}$ such that $I + (R \cap \mathcal{A}) = R$.*

On the other hand, if it is assumed furthermore that R is semilocal, then every finitely generated $R\pi$-projective module P is a free module.

Proof. By [CR, page 114, Theorem 5.24] $\operatorname{rad}(K\pi) = \sum_{\lambda \in \pi} K \cdot (\lambda - 1)$. Thus $K\pi/\operatorname{rad}(K\pi) \simeq K$. By Lemma 2.2 (with $I = \operatorname{rad}(K\pi)$), all the finitely generated $K\pi$-projective modules are free modules, because all the finitely generated projective modules over $K\pi/\operatorname{rad}(K\pi)$ ($\simeq K$) are the free modules $K^{(n)}$.

Consequently, if P is a finitely generated $R\pi$-projective module, then KP is a free $K\pi$-module. Thus we may apply Theorem 1.2 to P because the second assumption of Theorem 1.2 is valid by Theorem 1.3 (note that $\dim(m\text{-}\operatorname{spec}(R)) \leq 1$). Thus $P \simeq F \oplus \mathcal{A}$ where F is a free $R\pi$-module and $\mathcal{A}$ satisfies that, for any maximal ideal $\mathcal{M}$ of R, $\mathcal{A}/\mathcal{M}\mathcal{A}$ is isomorphic to $R'\pi$ where $R' = R/\mathcal{M}$. In case R is semilocal, then $\dim(m\text{-}\operatorname{spec}(R)) = 0$ and therefore finitely generated $R\pi$-projective modules are free by Theorem 1.2. We remark that the result when R is semilocal may be deduced also from Theorem 3.5.

From $P \simeq F \oplus \mathcal{A}$, we find that $KF \oplus K\mathcal{A} \simeq KP$ is $K\pi$-free. By the Krull-Schmidt-Azumaya's Theorem [CR, page 128] it follows that $K\mathcal{A} \simeq K\pi$. Thus $\mathcal{A}$ is a projective ideal of $R\pi$. It remains to show that $\mathcal{A}$ may be chosen such that $I + (R \cap \mathcal{A}) = R$ for any non-zero ideal I of R.

First we will show that $\mathcal{A}$ and the free module $R\pi$ belong to the same genus. For any maximal ideal $\mathcal{M}$ of R, consider the projective $R_{\mathcal{M}}\pi$-modules $\mathcal{A}_{\mathcal{M}}$ and $R_{\mathcal{M}}\pi$. As $\mathcal{M}R_{\mathcal{M}}\pi \subset \operatorname{rad}(R_{\mathcal{M}}\pi)$ by [La2, page 74, Corollary 5.9] and $\mathcal{A}_{\mathcal{M}}/\mathcal{M}\mathcal{A}_{\mathcal{M}} \simeq \mathcal{A}/\mathcal{M}\mathcal{A} \simeq R_{\mathcal{M}}\pi/\mathcal{M}R_{\mathcal{M}}\pi$, we may apply Lemma 3.2. It follows that $\mathcal{A}_{\mathcal{M}}$ and $R_{\mathcal{M}}\pi$ are isomorphic.

Once we know that $\mathcal{A}$ and $R\pi$ belong to the same genus, we may apply Roiter's Theorem [Sw3, page 37]. Thus we have an exact sequence of $R\pi$-modules $0 \to \mathcal{A} \to R\pi \to X \to 0$ such that $I + \mathrm{Ann}_R X = R$ where $\mathrm{Ann}_R X = \{r \in R : r \cdot X = 0\}$. Note that $R \cap \mathcal{A} = \mathrm{Ann}_R R\pi/\mathcal{A}$ and $R\pi/\mathcal{A} \simeq X$. Hence the result. ■

Remark. The assumption that no prime divisor of $|\pi|$ is a unit in R is crucial in the above Theorem 3.3 and in Theorem 1.1. In fact, if some prime divisor of $|\pi|$ is invertible in R, then $R\pi$ contains a non-trivial idempotent element (and thus KP will not be a free $K\pi$-module for some projective module P); Coleman shows that the converse is true also [CR, page 678].

The following theorem, due to S. Endo, provides an alternative proof of Theorem 3.3.

Theorem 3.4 *Let R be a Dedekind domain with* $\mathrm{char}\, R = p > 0$*, and π be a p-group. If P is a finitely generated $R\pi$-projective module, then P is isomorphic to $R\pi \otimes_R P_0$ for some R-projective module P_0, and is also isomorphic to a direct sum of a free module and a projective ideal of the form $R\pi \otimes_R I$ where I is some non-zero ideal of R. Moreover, for any non-zero ideal I' of R, the ideal I may be chosen so that $I + I' = R$.*

Proof. Let $\phi : R\pi \to R$ be the augmentation map defined by $\phi(\lambda) = 1$ for any $\lambda \in \pi$. Let J be the kernel of ϕ. Define $J_0 = \sum_{\lambda \in \pi} R \cdot (\lambda - 1)$. Then $J = J_0 \cdot R\pi$.

Let K be the quotient field of R. Then $\mathrm{rad}(K\pi) = J_0 \cdot K\pi$ by [CR, page 114]. Since $\mathrm{rad}(K\pi)$ is nilpotent, so is the ideal J of $R\pi$. It follows that $R\pi$ is J-complete and $J \subset \mathrm{rad}(R\pi)$.

Apply Lemma 2.2 to get a one-to-one correspondence of finitely generated projective modules over $R\pi$ and over R. For any finitely generated projective module P over $R\pi$, define $P_0 = P/JP$. Since both P and $R\pi \otimes_R P_0$ descend to P_0, it follows that P is isomorphic to $R\pi \otimes_R P_0$.

Every finitely generated projective R-module is isomorphic to $R^{(n)} \oplus I$ where n is a non-negative integer and I is a non-zero ideal of R (see [Sw3, page 219, Theorem A15]). Thus a finitely generated projective $R\pi$-module is isomorphic to a direct sum of a free module and a projective ideal of the form $R\pi \otimes_R I$. If I' is any non-zero ideal of R, we can find a non-zero ideal I_0 of R such that $I \simeq I_0$ and $I_0 + I' = R$ by [Sw3, page 218, Theorem A12]. ■

A corollary of Theorem 3.4 is the following.

Theorem 3.5 *Let R be a Dedekind domain with* $\mathrm{char}\, R = p > 0$*, and π be a p-group. Then R is a principal ideal domain if and only if every finitely generated $R\pi$-projective module is isomorphic to a free module.*

The following lemma is a partial generalization of Theorem 3.3 from p-groups to finite groups π with $p \mid |\pi|$.

Lemma 3.6 *Let R be a Dedekind domain with $\operatorname{char} R = p > 0$ and with quotient field K. Let π be a finite group such that $p \mid |\pi|$, and π_p be a p-Sylow subgroup of π. Let P be a finitely generated $R\pi$-projective module, P_{π_p} be the restriction of P to $R\pi_p$, and $(P_{\pi_p})^\pi := R\pi \otimes_{R\pi_p} (P_{\pi_p})$ be the induced module of P_{π_p}. Then $K(P_{\pi_p})^\pi$ is $K\pi$-free and $(P_{\pi_p})^\pi$ is isomorphic to $F \oplus \mathcal{A}$ where F is a free $R\pi$-module and $\mathcal{A}$ is a projective ideal of $R\pi$.*

Proof. If $K(P_{\pi_p})^\pi$ is $K\pi$-free, then we may apply Theorem 1.2 to finish the proof. It remains to show that $K(P_{\pi_p})^\pi$ is $K\pi$-free.

By Theorem 3.3, KP_{π_p} is $K\pi_p$-free. It follows that $K(P_{\pi_p})^\pi$ is $K\pi$-free. Done.

Note that P is a direct summand of $(P_{\pi_p})^\pi$ by [CR, pages 449-450]. ■

Example 3.7 A different proof of Theorem 1.1 other than that in [Sw1] is given in [Gr, Lecture 4]. It is proved first that, if R is a semilocal Dedekind domain with $\operatorname{char} R = 0$ and no prime divisor of $|\pi|$ is a unit in R, then every finitely generated $R\pi$-projective module is a free module [Gr, page 21,Theorem 4.7].

We remark that we may derive the above result directly from Theorem 1.1. For, if all the maximal ideals of R are $\mathcal{M}_1, \mathcal{M}_2, \ldots, \mathcal{M}_t$, define $I = \mathcal{M}_1 \cap \mathcal{M}_2 \cap \cdots \cap \mathcal{M}_t$ and apply Theorem 1.1. Then every finitely generated $R\pi$-projective module P is isomorphic to $F \oplus \mathcal{A}$ where F is free and $\mathcal{A}$ is a projective ideal of $R\pi$ with $I + (R \cap \mathcal{A}) = R$. It follows that $R \cap \mathcal{A} = R$, i.e. $1 \in \mathcal{A} \subset R\pi$. Thus $\mathcal{A} = R\pi$ is also a free module.

Note that, when $\pi = \{1\}$ is the trivial group, the similar statement as the above result (for semilocal rings) is not true in general. It is well-known that projective modules over a quasi-local ring are free modules (Kaplansky's Theorem; see [Sw2, page 82, Corollary 2.14] for the case of finitely generated projective modules).

When R is a commutative ring with only finitely many maximal ideals (e.g. a semilocal ring) having no non-trivial idempotent elements, then every projective R-module (which may not be finitely generated) is a free module, an analogy of Kaplansky's Theorem proved by Hinohara [Hi]; a similar result for finitely generated R-projective modules was proved independently by S. Endo.

Thus if R is a commutative ring with only finitely many maximal ideals, say, t is the number of distinct maximal ideals, we will show that there are at most t primitive idempotents in R. Write $R/\operatorname{rad}(R) = \prod_{1 \le i \le t} K_i$ where each K_i is a field (and is indecomposable). If $R = \prod_{1 \le j \le s} R_j$, from $\operatorname{rad}(R) = \prod_{1 \le j \le s} \operatorname{rad}(R_j)$, we find that $R/\operatorname{rad}(R)$ has at least s maximal ideals and therefore $s \le t$. Thus we may write $R = \prod_{1 \le j \le s} R_j$ where each R_j has no non-trivial idempotent elements; obviously $s \le t$.

Although a projective R-module is not necessarily free, it is isomorphic to a direct sum of free modules over these R_j's by applying Hinohara's Theorem.

Example 3.8 We remind the reader that Theorem 3 in [Ba1, page 533] is generalized as Theorem 8.2 in [Ba2, page 24] (see also [Sw2, page 171, Theorem 11.2]). We reproduce these two theorems as follows.

Theorem A ([Ba1, Theorem 3]) *Let R be a commutative noetherian ring, A be an R-algebra which is a finitely generated R-module and $d = \dim(m\text{-}\operatorname{spec}(R))$. Let P be a finitely generated A-projective module such that there is an integer r such that $P/\mathcal{M}P \simeq (A/\mathcal{M}A)^{(r)}$ for all maximal ideals $\mathcal{M}$ in R, then $P \simeq F \oplus Q$ where F is a free module of rank r', $Q/\mathcal{M}Q \simeq (A/\mathcal{M}A)^{(d')}$ for all maximal ideals $\mathcal{M}$ in R with $d' = \min\{d, r\}$ and $r' = r - d'$.*

Theorem B ([Ba2, page 24, Theorem 8.2]) *Let R, A, d be the same as above. Let P be a finitely generated A-projective module such that $P_{\mathcal{M}}$ contains a direct summand isomorphic to $A_{\mathcal{M}}^{(d+1)}$ for all maximal ideals $\mathcal{M}$ in R. Then $P \simeq A \oplus Q$ for some projective module Q.*

Let P be a finitely generated A-projective module in Theorem A. Note that the assumption for P (in the above Theorem A and also in Theorem 1.2) that $P/\mathcal{M}P \simeq (A/\mathcal{M}A)^{(r)}$ for all maximal ideals $\mathcal{M}$ in R is equivalent to the assumption that P and $A^{(r)}$ are locally isomorphic, i.e. $P_{\mathcal{M}} \simeq (A_{\mathcal{M}})^{(r)}$ for any maximal ideal $\mathcal{M}$ in R. The proof is the same as that in Theorem 3.3 for the projective ideal $\mathcal{A}$. Thus P satisfies the assumption of Theorem B.

When $r \geq d+1$, we find $P \simeq A \oplus Q$ by Theorem B. Since $A_{\mathcal{M}}$ is a (non-commutative) semilocal ring, the cancelation law is valid for finitely generated projective $A_{\mathcal{M}}$-modules [Sw2, page 176]. Thus $Q_{\mathcal{M}} \simeq A_{\mathcal{M}}^{(r-1)}$ for any maximal ideal $\mathcal{M}$ in R. Proceed by induction on r to obtain the conclusion of Theorem A.

§4. A local criterion

Finally we will discuss the following question. Let R be a commutative noetherian ring with total quotient ring K, A be an R-algebra which is a finitely generated projective R-module. Let P and Q be finitely generated A-projective modules. If $KP \simeq KQ$, under what situation, can we conclude that $P \simeq Q$?

The prototype of this question is a theorem of Brauer and Nesbitt [BN1, page 12, Theorem 2; CR, page 424, Corollary 17.10]: Let $(R, \mathcal{M})$ be a discrete valuation ring with quotient field K such that $\mathrm{char}(R/\mathcal{M}) = p > 0$. If π is a finite group, M and N are $R\pi$-lattices with $KM \simeq KN$, then $[M/\mathcal{M}M] = [N/\mathcal{M}N]$ in $G_0(R/\mathcal{M}\pi)$. A generalization of this theorem by Swan is given in [Sw1, Corollary 6.5]; see [CR, page 436, Corollary 18.16] also.

The above results of Brauer-Nesbitt and Swan are generalized furthermore by Bass as follows.

Theorem 4.1 (Bass [Ba1, Theorem 2; Sw3, page 12, Theorem 1.10; CR, page 671]) *Let $(R, \mathcal{M})$ be a local ring with total quotient ring K, A be an R-algebra which is a finitely generated R-projective module. Assume that the Cartan map $c : K_0(A/\mathcal{M}A) \to G_0(A/\mathcal{M}A)$ is injective. If P and Q are finitely generated A-projective modules such that $KP \simeq KQ$, then $P \simeq Q$.*

Theorem 4.2 *Let R be a commutative noetherian ring with total quotient ring K, A be an R-algebra which is a finitely generated R-projective module. Suppose that I is an ideal of R such that R/I is artinian. Let $\{\mathcal{M}_1, \ldots, \mathcal{M}_n\}$ be the set of all maximal ideals of R containing I. Assume that the Cartan map $c_i : K_0(A/\mathcal{M}_iA) \to G_0(A/\mathcal{M}_iA)$ is injective for all $1 \leq i \leq n$. If P and Q are finitely generated A-projective modules with $KP \simeq KQ$, then $P/IP \simeq Q/IQ$. Consequently, if R is semilocal and $I \subset \mathrm{rad}(R)$, then $P \simeq Q$.*

Proof. Step 1. Let $S = R \backslash \bigcup_{1 \leq i \leq n} \mathcal{M}_i$, $J = \bigcap_{1 \leq i \leq n} \mathcal{M}_i$. Then $S^{-1}R$ is a semilocal ring with maximal ideals $S^{-1}\mathcal{M}_1, S^{-1}\mathcal{M}_2, \ldots, S^{-1}\mathcal{M}_n$. Consider the projective modules $S^{-1}P$ and $S^{-1}Q$ over the algebra $S^{-1}A$. We will show that $S^{-1}P/S^{-1}JP \simeq S^{-1}Q/S^{-1}JQ$ in Step 2. Assume this result (which will be proved in Step 2). Then we apply Lemma 3.2 (note that $S^{-1}JA \subset \mathrm{rad}\ (S^{-1}A)$ by [La2, page 74, Corollary 5.9]). We get $S^{-1}P \simeq S^{-1}Q$, and therefore $S^{-1}P/S^{-1}IP \simeq S^{-1}Q/S^{-1}IQ$.

Write the primary decomposition of I as $I = \bigcap_{1 \leq i \leq n} I_i$ where I_i is an $\mathcal{M}_i$-primary ideal. Then $S^{-1}I = \bigcap_{1 \leq i \leq n} S^{-1}I_i$. For $1 \leq i \leq n$, since $\langle S, I_i \rangle = R$, it follows that $S^{-1}(R/I_i) \simeq R/I_i$. Thus $S^{-1}(A/I_iA) \simeq A/I_iA$ and $S^{-1}(P/I_iP) \simeq P/I_iP$, $S^{-1}(Q/I_iQ) \simeq Q/I_iQ$. Since $R/I \simeq \prod_{1 \leq i \leq n} R/I_i$, we get $P/IP \simeq \bigoplus_{1 \leq i \leq n} P/I_iP$, $S^{-1}P/S^{-1}IP \simeq \bigoplus_{1 \leq i \leq n} S^{-1}P/S^{-1}I_iP$ and similarly for Q and $S^{-1}Q$.

Now we have $P/IP \simeq \bigoplus_{1 \leq i \leq n} P/I_iP \simeq \bigoplus_{1 \leq i \leq n} S^{-1}(P/I_iP) \simeq S^{-1}P/S^{-1}IP$ and $Q/IQ \simeq S^{-1}Q/S^{-1}IQ$. Because we have shown that $S^{-1}P/S^{-1}IP \simeq S^{-1}Q/S^{-1}IQ$, we find that $P/IP \simeq Q/IQ$. If R is semilocal with $I \subset \mathrm{rad}(R)$, then $P \simeq Q$ by Lemma 3.2.

In summary, define $S = R\backslash\bigcup_{1\le i\le n}\mathcal{M}_i$, $J = \bigcap_{1\le i\le n}\mathcal{M}_i$ and consider the $S^{-1}A$-projective modules $S^{-1}P$ and $S^{-1}Q$. In the next paragraph, we will show that the assumption $KP \simeq KQ$ carries over to the ring $S^{-1}A$.

Let K_S be the total quotient ring of $S^{-1}R$ and let $\phi : R \to S^{-1}R$ be the canonical ring homomorphism. For any element $a \in R$, if a is not a zero-divisor, then $\phi(a)$ is not a zero-divisor in $S^{-1}R$. Thus the map ϕ may be extended to $K \to K_S$. It follows that $K_S \otimes_{S^{-1}R} S^{-1}P \simeq K_S \otimes_R P \simeq K_S \otimes_K KP$. Similarly, $K_S \otimes_{S^{-1}R} S^{-1}Q \simeq K_S \otimes_K KQ$. Since $KP \simeq KQ$ by assumption, it follows that $K_S \otimes_{S^{-1}R} S^{-1}P$ is also isomorphic to $K_S \otimes_{S^{-1}R} S^{-1}Q$.

It remains to prove that $S^{-1}P/S^{-1}JP \simeq S^{-1}Q/S^{-1}JQ$.

Step 2. To simplify the notation, we may assume, without loss of generality, that R is a semilocal ring with maximal ideals $\mathcal{M}_1, \ldots, \mathcal{M}_n$ and $J = \bigcap_{1\le i\le n}\mathcal{M}_i$. Let K be the total quotient ring of R. If $KP \simeq KQ$, we will prove that $P/JP \simeq Q/JQ$.

Define $S_i = R\backslash\mathcal{M}_i$ for $1 \le i \le n$. Let the total quotient ring of $S_i^{-1}R$ be K_i and $\phi_i : R \to S_i^{-1}R$ be the canonical ring homomorphism. As in the last two paragraph of Step 1, the map ϕ_i may be extended to a map $K \to K_i$ and we obtain an isomorphism of $K_i \otimes_{S_i^{-1}R} S_i^{-1}P$ with $K_i \otimes_{S_i^{-1}R} S_i^{-1}Q$.

Now we may apply Theorem 4.1 to the projective modules $S_i^{-1}P$ and $S_i^{-1}Q$ over the algebra $S_i^{-1}A$ for $1 \le i \le n$.

We find that $S_i^{-1}P \simeq S_i^{-1}Q$. Thus $S_i^{-1}P/S_i^{-1}JP \simeq S_i^{-1}Q/S_i^{-1}JQ$ for $1 \le i \le n$.

The remaining proof is analogous to that in Step 1. Note that $R/\mathcal{M}_i \simeq S_i^{-1}(R/\mathcal{M}_i)$. Thus $P/JP \simeq \bigoplus_{1\le i\le n} P/\mathcal{M}_iP \simeq \bigoplus_{1\le i\le n} S_i^{-1}P/S_i^{-1}\mathcal{M}_iP \simeq \bigoplus_{1\le i\le n} S_i^{-1}P/S_i^{-1}JP$ $\simeq \bigoplus_{1\le i\le n} S_i^{-1}Q/S_i^{-1}JQ \simeq \cdots \simeq Q/JQ$. ■

Remark. When R is a semilocal ring and A is a maximal R-order, an analogous result of Theorem 4.1 can be found in [Sw3, page 102, Corollary].

The following theorem is communicated to us by S. Endo. It provides a generalization of Theorem 1.4 (with the aid of Theorem 1.3).

Theorem 4.3 *Let $(R, \mathcal{M})$ be a commutative artinian local ring, A be an R-algebra which is a finitely generated free R-module. Then the Cartan map $K_0(A) \to G_0(A)$ is injective if and only if so is the Cartan map $K_0(A/\mathcal{M}A) \to G_0(A/\mathcal{M}A)$.*

Proof. Since R satisfies the ACC condition and the DCC condition on ideals, we can find a filtration of ideals of R as follows : $R = J_0 \supset J_1 \supset \ldots \supset J_t = 0$ where t is some positive integer and $J_{i-1}/J_i \simeq R/\mathcal{M}$ for $1 \le i \le t$.

As A is a free R-module, every finitely generated A-projective module P is also R-free. Tensor the exact sequence $0 \to J_i \to J_{i-1} \to R/\mathcal{M} \to 0$ with P over R. Note

that $J_i \otimes_R P \simeq J_i P$ as A-modules (because we may tensor the injection $0 \to J_i \to R$ with P). It follows that we obtain a filtration of A-modules $P = P_0 \supset P_1 = J_1 P \supset \ldots \supset P_t = J_t P = 0$ where $P_{i-1}/P_i \simeq P/\mathcal{M}P$. We conclude that $[P] = t[P/\mathcal{M}P]$ in $G_0(A)$.

By Lemma 2.3, find projective A-modules $P_1, P_2, \ldots, P_n$ and simple A-modules $M_1, M_2, \ldots, M_n$ such that $K_0(A) = \bigoplus_{1 \le i \le n} \mathbb{Z} \cdot [P_i]$ and $G_0(A) = \bigoplus_{1 \le i \le n} \mathbb{Z} \cdot [M_i]$. The same simple A-modules M_i satisfies that $G_0(A/\mathcal{M}A) = \bigoplus_{1 \le i \le n} \mathbb{Z} \cdot [M_i]$. Moreover, $K_0(A/\mathcal{M}A) = \bigoplus_{1 \le i \le n} \mathbb{Z} \cdot [P_i/\mathcal{M}P]$ by Lemma 2.2.

Now if $[P_i/\mathcal{M}P_i] = \sum_{1 \le j \le n} a_{ij}[M_j]$ in $G_0(A/\mathcal{M}A)$ where a_{ij} are some integers, then $[P_i] = \sum_{1 \le j \le n} t a_{ij}[M_j]$ in $G_0(A)$ (note that $G_0(A/\mathcal{M}A)$ is naturally isomorphic to $G_0(A)$ by [Sw2, page 94, Theorem 3.4]). Thus the determinant of the Cartan matrix $(a_{ij})_{1 \le i,j \le n}$ is non-zero if and only if so is that of $(t a_{ij})_{1 \le i,j \le n}$. ■

The following theorem of Rim is a generalization of Theorem 4.1. However, its proof was omitted in [Ri2]. For the convenience of the readers, we supply a proof of it as an application of Theorem 4.2 and Theorem 4.3.

Theorem 4.4 (Rim [Ri2, Theorem 7]) *Let R be a commutative noetherian ring with total quotient ring K, A be an R-algebra which is a finitely generated R-projective module. Suppose that I is an ideal of R such that R/I is artinian. Assume that the Cartan map $c : K_0(A/IA) \to G_0(A/IA)$ is injective. If P and Q are finitely generated A-projective modules with $KP \simeq KQ$, then $P/IP \simeq Q/IQ$.*

Proof. Write the primary decomposition of I as $I = \bigcap_{1 \le i \le n} I_i$ where I_i is an $\mathcal{M}_i$-primary ideal and each $\mathcal{M}_i$ is a maximal ideal of R. Then $A/IA \simeq \prod_{1 \le i \le n} A/I_i A$. It follows that this isomorphism induces isomorphisms $K_0(A/IA) \simeq \oplus_{1 \le i \le n} K_0(A/I_i A)$ and $G_0(A/IA) \simeq \oplus_{1 \le i \le n} G_0(A/I_i A)$. Note that, for $1 \le i \le n$, $A/I_i A$ is a R/I_i-free module and the Cartan map $K_0(A/I_i A) \to G_0(A/I_i A)$ is injective. Apply Theorem 4.3. We find that the Cartan map $K_0(A/\mathcal{M}_i A) \to G_0(A/\mathcal{M}_i A)$ is injective. Now we may apply Theorem 4.2 to finish the proof. ■

Example 4.5 With the aid of Theorem 4.1 we will show that Theorem B of Example 3.8 implies Theorem 1.2. Let A, R and d be given as in Theorem 1.2 and P be a finitely generated A-projective module. Suppose KP is free of rank r. For any maximal ideal $\mathcal{M}$ in R, consider $P_{\mathcal{M}}$. Now the (new!) base ring is the local ring $R_{\mathcal{M}}$. We will compare $P_{\mathcal{M}}$ with $P' = A_{\mathcal{M}}^{(r)}$.

Let $\phi : R \to R_{\mathcal{M}}$ be the canonical ring homomorphism, and let $K_{\mathcal{M}}$ be the total quotient ring of $R_{\mathcal{M}}$. For any element $a \in R$, if a is not a zero-divisor, then $\phi(a)$ is not a zero-divisor in $R_{\mathcal{M}}$. Thus the map ϕ may be extended to $K \to K_{\mathcal{M}}$. It follows

that $K_{\mathcal{M}} \otimes_{R_{\mathcal{M}}} P_{\mathcal{M}} \simeq K_{\mathcal{M}} \otimes_R P \simeq K_{\mathcal{M}} \otimes_K KP$ is a free module and $K_{\mathcal{M}} \otimes_{R_{\mathcal{M}}} P_{\mathcal{M}}$ is isomorphic to $K_{\mathcal{M}} \otimes_{R_{\mathcal{M}}} P'$.

Apply Theorem 4.1. We find that $P_{\mathcal{M}} \simeq P' = A_{\mathcal{M}}^{(r)}$. Since P is locally free, we may apply Theorem B of Example 3.8 so that $P \simeq A \oplus Q$ where Q is locally free of rank $r-1$ if $r \geq d+1$ as in Example 3.8. The proof of Theorem 1.2 is finished by induction on r.

In general, a finitely generated $R\pi$-projective module may be written as a direct sum of indecomposable $R\pi$-projective modules. The following lemma tells what an indecomposable $R\pi$-projective module looks like in case $\mid \pi \mid$ is invertible in R.

Lemma 4.6 *Let R be a Dedekind domain with quotient field K, π be a finite group such that $|\pi|$ is invertible in R. If P is a finitely generated indecomposable $R\pi$-projective module, then P is isomorphic to a projective ideal of $R\pi$; moreover, there is some projective ideal $\mathcal{A}$ generated by a primitive idempotent of $R\pi$ such that P and $\mathcal{A}$ belong to the same genus.*

Proof. Note that $R\pi$ becomes a maximal R-order because $|\pi|$ is invertible in R [CR, page 582]. As such, it is known that (i) $R\pi$ is left hereditary; (ii) a finitely generated $R\pi$-module P is $R\pi$-projective if and only if it is an $R\pi$-lattice; (iii) the module P is an indecomposable $R\pi$-projective module if and only if KP is a simple $K\pi$-module [CR, page 565].

Now we come to the proof. By a theorem of Kaplansky every projective module over a left hereditary ring is a direct sum of projective ideals (see [CE, page 13, Theorem 5.3]). Since the projective module P we consider is indecomposable, it is isomorphic to a projective ideal of $R\pi$. It remains to find some projective ideal $\mathcal{A}$ such that $\mathcal{A}$ is a direct summand of $R\pi$ satisfying that P and $\mathcal{A}$ belong to the same genus.

Since KP is a simple $K\pi$-module, it is isomorphic to a minimal left ideal V of $K\pi$ by the Artin-Wedderburn Theorem. Since $K\pi$ is semi-simple, write $K\pi = V \oplus V'$ where V' is another left ideal of $K\pi$.

From the embedding $R\pi \to K\pi$, define $\mathcal{A} = R\pi \cap V$ and define $\mathcal{A}'$ by the exact sequence $0 \to \mathcal{A} \to R\pi \to \mathcal{A}' \to 0$. Hence $K\mathcal{A} = V$ and $\mathcal{A}'$ is R-torsion free. It follows that $\mathcal{A}'$ is $R\pi$-projective and the exact sequence $0 \to \mathcal{A} \to R\pi \to \mathcal{A}' \to 0$ splits. Thus $\mathcal{A}$ is generated by an idempotent element u of $R\pi$. This idempotent element u is primitive because $\mathcal{A}$ is indecomposable (remember that $K\mathcal{A} = V$ which is a simple $K\pi$-module).

Note that, if $i : P \to R\pi \cap V$ $(= \mathcal{A})$ is the embedding of P via Kaplansky's Theorem and $KP \simeq V$ (see the proof of [CE, page 13, Theorem 5.3]), it is not true in general that $i(P)$ should be equal to $\mathcal{A}$.

Finally we will show that P and $\mathcal{A}$ belong to the same genus. Both KP and $K\mathcal{A}$ are isomorphic to V. Because $R\pi$ is a maximal order, we may apply [CR, page 643, Proposition 31.2] to finish the proof. Note that this result may be proved alternatively by applying Theorem 4.1. ■

Remark. In the above lemma, $K\mathcal{A}$ is not free if $|\pi| > 1$. In case KP is free, the following result is known: Let R be a Dedekind domain with quotient field K and π be a finite group. If $\gcd\{|\pi|, \operatorname{char} R\} = 1$ and P is a finitely generated $R\pi$-projective module such that KP is free, then $P \simeq F \oplus \mathcal{A}$ where F is a free module and $\mathcal{A}$ is some projective ideal (see [Sw1, Theorem 7.2]).

Lemma 4.7 (Villamayor [Vi]) *Let π be a finite group and R be a commutative ring such that rad$(R) = 0$ and $|\pi|$ is a unit in R. Then rad$(R\pi) = 0$.*

Proof. This theorem is proved essentially in [Vi, page 626, Theorem 3]; as noted in [Vi, page 627, Remark 1], if R is a commutative ring such that $|\pi|$ is a unit in R, the proof of Theorem 3 in [Vi, page 626] remains valid (as π is a finite group). Be aware that, according to the convention of [Vi, page 621], a ring A is called semisimple if $\operatorname{rad}(A) = 0$. Villamayor's Theorem can be found also in [Pa, page 278]; it is easy to check that the proof of this theorem in [Pa, page 278] works as well so long as R is any commutative ring such that $|\pi|$ is a unit in R (in other words, the assumption that R is a field may be relaxed). ■

Example 4.8 Let π be a finite group. Choose a Dedekind domain R such that R is not semilocal and $|\pi|$ is a unit in R. Then $\operatorname{rad}(R) = 0$. Thus $\operatorname{rad}(R\pi) = 0$ by Villamayor's Theorem. It follows that $R\pi/\operatorname{rad}(R\pi) \simeq R\pi$ is not left artinian. Hence $R\pi$ is not semiperfect [La2, page 346]. By Theorem (25.3) of [La2, page 371], any finitely generated indecomposable projective module over a semiperfect ring is isomorphic to a projective ideal generated by a primitive idempotent (compare this result with Lemma 4.6).

Lemma 4.9 *Let R be a commutative noetherian integral domain with* $\operatorname{char} R = p > 0$*, π be a finite group such that $p \mid |\pi|$. Assume that the p-Sylow subgroup π_p is normal in π. Write $\pi' = \pi/\pi_p$.*

(1) *Define a right ideal $I := \sum_{\lambda \in \pi_p} (\lambda - 1) \cdot R\pi$. Then I is a nilpotent two-sided ideal of $R\pi$, $R\pi/I \simeq R\pi'$, and rad$(R\pi) = \langle I, \operatorname{rad}(R) \rangle$.*

(2) *There is a one-to-one correspondence between the isomorphism classes of finitely generated $R\pi$-projective modules and the isomorphism classes of finitely generated $R\pi'$-projective modules given by $P \rightsquigarrow P/IP$ where I is defined in* (1)*. Note that $|\pi'|$ is a unit in R.*

Proof. Step 1. For any $\sigma \in \pi$ and any $\lambda \in \pi_p$, $\sigma(\lambda - 1)\sigma^{-1} \in I$, because π_p is a normal subgroup of π. Thus I is a two-sided ideal of $R\pi$. Clearly $R\pi/I \simeq R\pi'$.

Let K be the quotient field of R. As in the proof of Theorem 3.3, we find that $\mathrm{rad}(K\pi_p) = \sum_{\lambda \in \pi_p} K \cdot (\lambda - 1)$. Since $\mathrm{rad}(K\pi_p)$ is nilpotent, so is the ideal $I_0 := \sum_{\lambda \in \pi_p} R \cdot (\lambda - 1)$ in $R\pi_p$. It follows that $I = I_0 \cdot R\pi$ and $I^n = I_0^n \cdot R\pi$. Thus I is nilpotent and is contained in $\mathrm{rad}(R\pi)$. Note that $|\pi'|$ is a unit in R.

Using the fact that $|\pi'|$ is a unit in R, we will show that $\mathrm{rad}(R\pi') = \mathrm{rad}(R) \cdot R\pi'$. Because $\mathrm{rad}(R) \cdot R\pi' \subset \mathrm{rad}(R\pi')$, the fact that $\mathrm{rad}(R\pi') = \mathrm{rad}(R) \cdot R\pi'$ is equivalent to $\mathrm{rad}(R'\pi') = 0$ where $R' = R/\,\mathrm{rad}(R)$. The latter assertion is true by Lemma 4.7. Hence $\mathrm{rad}(R\pi') = \mathrm{rad}(R) \cdot R\pi'$.

From $\mathrm{rad}(R\pi/I) \simeq \mathrm{rad}(R\pi')$ and $\mathrm{rad}(R\pi/I) = \mathrm{rad}(R\pi)/I$ [La2, page 55], we find that $\mathrm{rad}(R\pi) = \langle I, \mathrm{rad}(R) \rangle$.

Step 2. Since I is nilpotent, $R\pi$ is I-complete. Apply Lemma 2.2 to get the one-to-one correspondence of finitely generated projective modules over $R\pi$ and $R\pi'$.

■

Remark. Let the notations be the same as the above lemma. Assume furthermore that the group extension $1 \to \pi_p \to \pi \to \pi' \to 1$ splits. Then the composite of the imbedding $R\pi' \to R\pi$ and the canonical projection $R\pi \to R\pi'$ is the identity map on $R\pi'$. By the same idea of Theorem 3.4, it can be shown that every finitely generated $R\pi$-projective module is of the form $R\pi \otimes_{R\pi'} P_0$ for some $R\pi'$-projective module P_0.

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